Malaysian Journal of Mathematical Sciences 19(4): 1335–1349 (2025)
https://doi.org/10.47836/mjms.19.4.09

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

On Laplacian Commutativity of Graphs

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Received: 6 February 2025
Accepted: 28 May 2025

Abstract

This paper introduces the notion of Laplacian commutativity of graphs among well known classes of graphs. Two graphs are Laplacian commutative if their Laplacian matrices commute. The commutativity of the Laplacian matrix of a graph G with its complement, G', and its $k-$complement, ${G_k}^{\mathcal{P}}$ is also examined. Laplacian commutativity depends on the partition $\mathcal{P}$ of vertex set of G, V_G and ${G_k}^{\mathcal{P}}$. Some necessary and sufficient conditions on the partition $\mathcal{P}$ are described for the Laplacian commutativity of cycle C_n with $(C_n)_k^{\mathcal{P}}$.

Keywords: Laplacian matrix; adjacency matrix; commutativity; graph complement; $k-$complement; $GH-$*path*.

1 Introduction

The Laplacian matrix, also referred to as the Kirchhoff matrix or discrete Laplacian, represents a graph mathematically and is named after Pierre–Simon Laplace. This matrix plays a significant role in various computational science fields, including graph isomorphism problems, electrical networks, machine learning, and signal processing. This paper focuses solely on non-directional finite graphs without self-loops and multiple edges. Throughout this paper, n denotes the number of vertices of the graph G, referred to as the order of the graph G. A graph $G = (V_G, E_G)$ is specified by its vertex set, $V_G = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set, $E_G = \{e_1, e_2, e_3, \dots, e_m\}$, where each e_i consists of a pair of vertices, with no specific order.

The adjacency matrix of G is the $n \times n$ matrix $A(G) = [a_{ij}]$ in which $a_{ij} = 1$, if v_i and v_j are adjacent, and 0 otherwise. The Laplacian matrix of G, $L(G)$, is the $n \times n$ matrix, where the $(i, j)^{th}$ entry, for $i \neq j$ is 0, if the vertices v_i and v_j are not adjacent, and -1 if v_i and v_j are adjacent, and the $(i, i)^{th}$ entry corresponds to the degree, $d_G(v_i)$, of the vertex v_i, for $i = 1, 2, 3, \dots, n$. The relationship between adjacency and Laplacian matrices provides important insights into graph structure and connectivity; for instance, West [8] elaborates on how these matrices encode vertex degrees and adjacency relations, offering tools to study graph connectivity, eigenvalues, and spanning trees. The complement of a graph G is the graph G' with vertex set $V_{G'} = V_G$, where two vertices in G' are adjacent if and only if they are not adjacent in G. The Laplacian matrix of a graph complement can be expressed in terms of the degree and adjacency matrices of the original graph.

These fundamental concepts form the base of graph theory and provide the foundation for studying the structural and spectral properties of graphs, their complements, and their partitions. We explore the commutativity of graphs via adjacency matrices and Laplacian matrices. For a comprehensive overview of graph-theoretic matrix concepts, including detailed treatments of adjacency and Laplacian matrices, Bapat [2] offers foundational results that support the spectral analysis techniques used in this study.

This work focuses on the concept of commutativity of graphs, which is formally described and analyzed through the Laplacian matrix. For regular graphs, an interesting equivalence has been established: the adjacency matrix of a graph and its complement commute with one another if and only if the Laplacian matrix of the graph and its complement commute. This equivalence forms a foundational aspect of the present study, providing a base for further exploration of the commutative relationships between graphs, their complements, and $k-$complements. Furthermore, the Laplacian commutativity of Cycle graphs with its $k-$complement were studied and derived some properties of the partition $\mathcal{P}$ of their vertex sets for Laplacian commutativity.

2 Related Works

Akbari et al. [1] have defined graph commutativity using the commutativity of their adjacency matrices. Two graphs, G and H, are considered to commute if their adjacency matrices commute with each other, assuming they have an equal number of vertices.

To expand the notion of a graph's complement, Sampathkumar et al. [7] proposed the idea of the $k-$complement $G_k^{\mathcal{P}}$ and the $k(i)-$complement $G_{k(i)}^{\mathcal{P}}$ of a graph G with respect to a partition $\mathcal{P} = (V_1, V_2, V_3, \dots, V_k)$ of the vertex set of G. This study covers the the adjacency commutativity properties of some graphs with its $k-$complement $G_k^{\mathcal{P}}$. The $k-$complement of G, $G_k^{\mathcal{P}}$, $(k \geq 2)$ with respect to the partition $\mathcal{P} = (V_1, V_2, V_3, \ldots, V_k)$ is the graph constructed on the vertex set, V_G in such a way that two vertices $u_i, v_i \in V_i$ are adjacent in $G_k^{\mathcal{P}}$ if and only if they are adjacent in G and two vertices $u_i \in V_i$ and $u_j \in V_j$, for $i \neq j$ are adjacent in $G_k^{\mathcal{P}}$ if and only if they are not adjacent in G.

The out degree of a vertex v, called $o-degree$ of v in a vertex set V_i, $i = 1, 2, \ldots, k$ with respect to a partition $\mathcal{P} = (V_1, V_2, V_3, \ldots, V_k)$ of V_G is defined in [3] as the number of vertices in $V_G \backslash V_i$ that are adjacent to v in G.

Manjunathaprasad et al. [5] introduced the concept of $GH-path$ as: Given graphs G and H on the same set of vertices $\{v_1, v_2, v_3, \ldots, v_n\}$, two vertices v_i and v_j, are said to have a $GH-path$ from v_i to v_j, if there exists a vertex v_k, different from v_i and v_j such that v_i adjacent to v_k in G and v_k is adjacent to v_j in H and this $GH-path$ through v_k is denoted as: $v_i \sim_G v_k \sim_H v_j$. For convenience, we will denote such a $GH-path$ as, $v_i \sim v_k \sim v_j$.

Bhat et al. [3] derived some properties of partition $\mathcal{P}$ of V_G of size $k < n$, with n being the number of vertices in G, for which the adjacency matrix $A(G)$, commutes with $A(G_k^{\mathcal{P}})$. The graph G and the partition $\mathcal{P}$ were described in a manner that ensures G commutes with $G_k^{\mathcal{P}}$. By bridging the gap between adjacency and Laplacian commutativity, this work contributes to a deeper understanding of the algebraic interplay between graphs and their complements.

The study by Romdhini et al. [6] presents an adjacency-based spectral approach to analyzing commuting and non-commuting graphs through the Neighbors Degree Sum Energy (NDSE). Focusing on dihedral groups, their work demonstrates how energy-based metrics can reflect structural and algebraic characteristics of graphs. While their approach is rooted in adjacency properties, our study complements this perspective by examining commutativity through Laplacian matrices and generalized complements derived from vertex partitions.

3 Results

3.1 Laplacian commutativity of graphs

We now define the Laplacian commutativity of two graphs as follows.

Definition 3.1. *Two graphs G_1 and G_2 are said to satisfy Laplacian commutativity if and only if the Laplacian matrices $L(G_1)$ and $L(G_2)$ commute, i.e,*

$$L(G_1)L(G_2) = L(G_2)L(G_1). \tag{1}$$

To avoid ambiguity, we may use the term adjacency commutativity for the graph commutativity defined by Akbari et al. [1]. They investigated such commutativity in the context of $K_{n,n}$ identifying when it can be decomposed into commuting perfect matchings or Hamiltonian cycles. They also determine the centralizers of specific families of graphs, highlighting deeper algebraic structures behind commuting adjacency matrices.

Example 3.1. *Let S_n denote star graph and W_n represent the wheel graph with the set of vertices $\{v_1, v_2, v_3, \ldots, v_n\}$. If the vertices are ordered in such a way that the central vertex of S_n aligns with the central vertex of W_n, then, the Laplacian matrices of S_n and W_n always commute. This commutativity* *arises because the vertex alignments ensure structural compatibility in the matrix operations. Conversely, when the central vertices do not coincide in the order, the correspondence between the graph structures is disrupted, resulting in a lack of commutativity. Here, the Laplacian matrices of S_n and W_n, where v_1 as the central vertex are given,*

$$L(S_n) = \begin{bmatrix} n-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and

$$L(W_n) = \begin{bmatrix} n-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & 3 & -1 & 0 & \cdots & -1 \\ -1 & -1 & 3 & -1 & \cdots & 0 \\ -1 & 0 & -1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & 0 & \cdots & 3 \end{bmatrix}.$$

The products, $L(S_n)L(W_n)$ and $L(W_n)L(S_n)$ are equal and given as follows,

$$L(S_n)L(W_n) = L(W_n)L(S_n) = \begin{bmatrix} n(n-1) & -(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1) \\ -(n-1) & 4 & 0 & 1 & \cdots & 0 \\ -(n-1) & 0 & 4 & 0 & \cdots & 1 \\ -(n-1) & 1 & 0 & 4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n-1) & 0 & 1 & 1 & \cdots & 4 \end{bmatrix}. \tag{2}$$

It is recognized that the adjacency matrix of a regular graph G and the adjacency matrix of its complement graph G' commute with each other. This commutativity property holds for regular graphs because of the uniform degree of vertices, which ensures symmetry in the graph's structure, making the adjacency matrices and their complements commute.

3.2 Laplacian commutativity of graphs with its complement

In this section, we explore the conditions under which a graph G and its complement G' exhibit commutativity with respect to the adjacency matrix $A(G)$ and Laplacian matrix $L(G)$. We characterize adjacency commutativity by showing that it holds if and only if $A(G)$ commutes with the matrix J, an $n \times n$ matrix with all entries equal to 1, which occurs precisely when G is regular. We then show that, unlike adjacency matrices, Laplacian matrices of a graph and its complement always commute, regardless of the graph's regularity.

Proposition 3.1. *Let G be a graph on n vertices and G' be its complement. Then, $A(G)$ and $A(G')$ are adjacency commutative if and only if its adjacency matrix, $A(G)$ commute with J.*

Proof. Let $A(G)$ denote the adjacency matrix of a given graph G, of size $n \times n$. Let I represent the identity matrix of order n. Additionally, let K_n be the complete graph of n vertices.

Then, $A(G)$ and $A(G')$ are adjacency commutative if and only if,

$$A(G)A(G') = A(G')A(G).$$

Two vertices in G' are adjacent if and only if they are not adjacent in G. Hence,

$$A(G) + A(G') = A(K_n) = J - I,$$

which implies that,

$$A(G') = J - A(G) - I.$$

Multiplying $A(G)$ and $A(G')$, we obtain,

$$\begin{aligned} A(G)A(G') &= A(G)(J - A(G) - I) \\ &= A(G)J - A^2(G) - A(G), \\ A(G')A(G) &= JA(G) - A^2(G) - A(G). \end{aligned}$$

Thus, G and G' are adjacency commutative if and only if $A(G)J = JA(G)$. Hence, we proved Proposition 3.1. □

Theorem 3.1. *A graph G and its complement G' are adjacency commutative if and only if the graph G is regular; that is, all vertices of G have the same degree.*

Proof. Now, the $(i,k)^{th}$ entry of $A(G)J$ is calculated as follows,

$$\begin{aligned} (A(G)J)_{ik} &= \sum_{j=1}^{n} a_{ij}.1 \\ &= \sum_{j=1}^{n} a_{ij}, \end{aligned}$$

and the $(i,k)^{th}$ entry of $JA(G)$ is computed as,

$$\begin{aligned} (JA(G))_{ik} &= \sum_{j=1}^{n} a_{jk}.1 \\ &= \sum_{j=1}^{n} a_{jk}. \end{aligned}$$

Hence, $A(G)J = JA(G)$ if and only if all vertices are of same degree. □

For non-regular graphs, it may happen that the adjacency commutativity of G and G' does not hold, but the Laplacian commutativity does. An example of such a graph is the wheel graph of order 5, denoted W_5, for which the Laplacian matrices $L(G)$ and $L(G')$ commute, despite G being non-regular and $A(G)A(G') \neq A(G')A(G)$. In fact, Laplacian commutativity holds for any simple connected graphs with its complement.

Theorem 3.2. *The Laplacian matrix of any simple undirected graph always commute with its complement.*

Proof. The Laplacian matrices of G and G' are given by,

$$L(G) = D(G) - A(G) \text{ and } L(G') = D(G') - A(G'),$$

where $D(G)$ denotes the diagonal matrix of the degree of vertices of the graph G. Let the degree of a vertex, v, in a graph, G, be denoted by $d_G(v)$, and let the degree of v, in its complement, G', be denoted by $d_{G'}(v)$.

In a simple graph with n vertices, each vertex can be adjacent to at most $n-1$ vertices and by the definition of the complement of a graph it can be obtained that, two vertices are adjacent in G if and only if they are not adjacent in G'. Therefore, the adjacent vertices of G' are precisely given by, $d_{G'}(v) = n - 1 - d_G(v)$. Then, we have,

$$D(G') = nI - J - D(G), \text{where } I \text{ denote identity matrix of order } n.$$

Thus, the Laplacian matrices $L(G)$ and $L(G')$ are related by the equation,

$$L(G) + L(G') = nI - J.$$

Hence,

$$\begin{aligned} L(G)L(G') &= L(G)(nJ - J - L(G)) \\ &= nL(G) - L(G)J - L(G)^2 \\ &= nL(G) - L(G)^2, \quad \text{as} \quad L(G)J = 0. \end{aligned}$$

Similarly, $L(G')L(G) = nL(G) - L(G)^2$, which implies that, $L(G)L(G') = L(G')L(G)$. □

Another pair of commuting graphs in the sense of Laplacian commutativity is the complete graph, K_n, with any of its spanning sub-graph T, which is defined as a subgraph that contains all the vertices of the original graph and a subset of its edges.

Theorem 3.3. *The complete graph K_n and any other graph G defined on an identical set of vertices exhibit Laplacian commutativity.*

Proof. The Laplacian matrix of a Complete graph on n vertices, K_n can be written as,

$$K_n = nI_n - J_n.$$

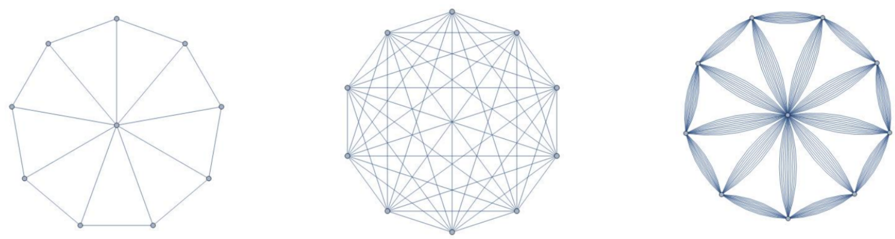
Hence, for any graph G,

$$L(K_n)L(G) = nL(G) = L(G)L(K_n),$$

which completes the proof. □

It is important to note that, the product of two Laplacian matrices does not, in general, yield the Laplacian matrix of any standard graph product. From the above theorem we get the resulting matrix is the Laplacian matrix of a multi-graph. which is defined in [4] as: A multi-graph M_G consists of a finite non-empty set V_G of vertices and E_G of edges, where every two vertices of M_G are joined by a finite number of edges (possibly zero). In contrast to a simple graph, where each pair of vertices can be connected by at most one edge, a multi-graph permits the existence of more than one edge between two vertices.

Figure 1 demonstrates an example of the Laplacian graph corresponding to the product matrix of the Laplacian matrices of a complete graph, K_{10}, and a wheel graph, W_{10}. Here, by the Laplacian graph, we refer to the graph associated with the resulting Laplacian matrix.



(a) W_{10} (b) K_{10} (c) Laplacian graph of the product matrix

Figure 1: Graphical representation of W_{10}, K_{10} and the Laplacian graph of the product matrix.

From Theorem 3.3, the product is obtained as $L(W_{10})L(K_{10}) = 10L(K_{10})$, which is a complete multigraph on 10 vertices.

3.3 Laplacian commutativity of graphs with its $k-$complement

While investigating the Laplacian commutativity of a graph and its generalized $k-$complement with respect to a given partition $\mathcal{P}$ of its vertex set, we observe distinct behaviors across different classes of graphs. we could find certain types of graphs, such as complete graphs, where the result holds for any partition reflecting a strong structural regularity. In contrast, specific classes such as cycles demonstrate this property only for certain partitions and certain graphs such as some trees where the result fails regardless of the chosen partition. Based on preliminary observations, we conjecture that all trees fall into the third category. In the subsequent section, we explore the Laplacian commutativity of the complete graph with its $k-$complement in detail.

3.3.1 Complete graphs

Theorem 3.4. *Let G be the complete graph on n vertices. Then, G and the $k-$complement $G_k^{\mathcal{P}}$ satisfies Laplacian commutativity for any partition $\mathcal{P} = (V_1, V_2, V_3, \dots, V_k)$ of V_G, where $k \geq 2$.*

Proof. Consider $G = K_n$, the complete graph on n vertices and a partition $\mathcal{P} = (V_1, V_2, V_3, \dots, V_k)$ of V_G of order k. If $|V_i| = n_i$, $1 \leq i \leq k$, then, with an appropriate ordering of the vertices $L(G)$ can be written as,

$$L(G) = \begin{bmatrix} (nI-J)_{n_1\times n_1} & -J_{n_1\times n_2} & -J_{n_1\times n_3} & \cdots & -J_{n_1\times n_k} \\ -J_{n_2\times n_1} & (nI-J)_{n_2\times n_2} & -J_{n_2\times n_3} & \cdots & -J_{n_2\times n_k} \\ -J_{n_3\times n_1} & -J_{n_3\times n_2} & (nI-J)_{n_3\times n_3} & \cdots & -J_{n_3\times n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -J_{n_k\times n_1} & -J_{n_k\times n_2} & -J_{n_k\times n_3} & \cdots & (nI-J)_{n_k\times n_k} \end{bmatrix}.$$

Concerning the partition $\mathcal{P}$, the Laplacian matrix of the $k-$complement of $G_k^{\mathcal{P}}$ is given by,

$$L(G_k^{\mathcal{P}}) = \begin{bmatrix} (n_1I - J)_{n_1\times n_1} & 0_{n_1\times n_2} & 0_{n_1\times n_3} & \cdots & 0_{n_1\times n_k} \\ 0_{n_2\times n_1} & (n_2I - J)_{n_2\times n_2} & 0_{n_2\times n_3} & \cdots & 0_{n_2\times n_k} \\ 0_{n_3\times n_1} & 0_{n_3\times n_2} & (n_3I - J)_{n_3\times n_3} & \cdots & 0_{n_3\times n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{n_k\times n_1} & 0_{n_k\times n_2} & 0_{n_k\times n_3} & \cdots & (n_kI - J)_{n_k\times n_k} \end{bmatrix}.$$

Then, the product $L(G)L(G_k^{\mathcal{P}})$ is given by the block matrix,

$$L(G)L(G_k^{\mathcal{P}}) = \begin{bmatrix} ((nI-J)(n_1I-J))_{n_1\times n_1} & -J(n_2I-J)_{n_1\times n_2} & \cdots & -J(n_kI-J)_{n_1\times n_k} \\ -J(n_1I-J)_{n_2\times n_1} & ((nI-J)(n_2I-J))_{n_2\times n_2} & \cdots & -J(n_kI-J)_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ -J(n_1I-J)_{n_k\times n_1} & -J(n_2I-J)_{n_k\times n_2} & \cdots & ((nI-J)(n_kI-J))_{n_k\times n_k} \end{bmatrix}.$$

After simplification, the matrix becomes block diagonal. Hence,

$$L(G)L(G_k^{\mathcal{P}}) = \begin{bmatrix} ((nI-J)(n_1I-J))_{n_1\times n_1} & 0_{n_1\times n_2} & \cdots & 0_{n_1\times n_k} \\ 0_{n_2\times n_1} & ((nI-J)(n_2I-J))_{n_2\times n_2} & \cdots & 0_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_k\times n_1} & 0_{n_k\times n_2} & \cdots & ((nI-J)(n_kI-J))_{n_k\times n_k} \end{bmatrix}.$$

Similarly, the product $L(G_k^{\mathcal{P}})L(G)$ is the block diagonal matrix,

$$L(G_k^{\mathcal{P}})L(G) = \begin{bmatrix} ((n_1I-J)(nI-J))_{n_1\times n_1} & 0_{n_1\times n_2} & \cdots & 0_{n_1\times n_k} \\ 0_{n_2\times n_1} & ((n_2I-J)(nI-J))_{n_2\times n_2} & \cdots & 0_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_k\times n_1} & 0_{n_k\times n_2} & \cdots & ((n_kI-J)(nI-J))_{n_k\times n_k} \end{bmatrix}.$$

Hence,

$$L(G)L(G_k^{\mathcal{P}}) = L(G_k^{\mathcal{P}})L(G).$$

□

3.3.2 Cycles

Let C_n be a cycle graph on n vertices $\{v_1, v_2, v_3, \ldots, v_n\}$ with $v_i \sim_{C_n} v_{i+1}$, $i = 1, 2, 3, \ldots, n-1$, $v_n \sim_{C_n} v_1$, as defined in [3]. The $k-$complement of C_n with respect to a partition $\mathcal{P} = (V_1, V_2, \ldots, V_k)$ of $V(C_n)$ is denoted as $(C_n)_k^{\mathcal{P}}$. Here, we find some conditions for a partition of the vertex set $V(C_n)$ so that the Laplacian matrices of cycle graph C_n and $(C_n)_k^{\mathcal{P}}$ commute.

Lemma 3.1. *Let, C_n be the cycle graph on n vertices $\{v_1, v_2, v_3, \ldots, v_n\}$. Let, $\mathcal{P} = (V_1, V_2, V_3, \ldots, V_k)$ be a partition of the vertex set $V(C_n)$, and let $(C_n)_k^{\mathcal{P}}$ be the $k-$complement of C_n regarding the partition $\mathcal{P}$. The graphs C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative if and only if,*

$$l'_{i-1j} + l'_{i+1j} = l'_{ij-1} + l'_{ij+1}, \quad \textit{for all } i,\ j, \tag{3}$$

where l'_{ij} denotes the $(i, j)^{th}$ entry of $L((C_n)_k^{\mathcal{P}})$.

Proof. We have the i^{th} row in $L(C_n)$ is of the form $(0, \ldots, 0, -1, 2, -1, 0, \ldots, 0)$. Now, if the j^{th} column of $L(C_n)_k^{\mathcal{P}}$ is $(l'_{1j}, l'_{2j}, \ldots, l'_{ij}, \ldots, l'_{nj})^T$, then the $(i, j)^{th}$ entry of $L(C_n)L((C_n)_k^{\mathcal{P}})$ and $L((C_n)_k^{\mathcal{P}})L(C_n)$ are given by,

$$[L(C_n)L((C_n)_k^{\mathcal{P}})]_{ij} = -l'_{i-1j} + 2l'_{ij} - l'_{i+1j},$$
$$[L((C_n)_k^{\mathcal{P}})L(C_n)]_{ij} = -l'_{ij-1} + 2l'_{ij} - l'_{ij+1}.$$

So that C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative if and only if,

$$l'_{i-1j} + l'_{i+1j} = l'_{ij-1} + l'_{ij+1}, \quad \text{for all } i,\ j.$$

□

Remark 3.1. *From Lemma 3.1, it can be observed that if C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative, the entries in $L((C_n)_k^{\mathcal{P}})$ satisfies the above condition. In order to satisfy this condition, $L((C_n)_k^{\mathcal{P}})$ must have equal diagonal entries. (To be precise, the degree of each vertex must be equal.)*

Theorem 3.5. *Let $(C_n)_k^{\mathcal{P}}$ be the $k-$complement of C_n in relation to a partition $\mathcal{P} = (V_1, V_2, \ldots, V_k)$ of $V(C_n)$. If C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative, then for every two vertices v_i and v_j, the number of $C_n(C_n)_k^{\mathcal{P}}-$paths and $(C_n)_k^{\mathcal{P}}C_n-$paths from v_i to v_j are equal.*

Proof. Suppose that C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative. In a Cycle graph $C_n = v_1v_2 \ldots v_nv_1$, the vertices which are adjacent to vertex v_i are v_{i-1} and v_{i+1}. Hence, the number of $C_n(C_n)_k^{\mathcal{P}}$ and $(C_n)_k^{\mathcal{P}}C_n-$*paths* from v_i to v_j is at most 2, for any two vertices v_i and v_j.

Case (i): There is no $C_n(C_n)_k^{\mathcal{P}}$ and $(C_n)_k^{\mathcal{P}}C_n-$*paths* from v_i to v_j.
Assume that, there does not exist $C_n(C_n)_k^{\mathcal{P}}-$*path* from v_i to v_j. Since C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative, By Lemma 3.1,we have,

$$l'_{i-1j} + l'_{i+1j} = l'_{ij-1} + l'_{ij+1}, \quad \text{for all } l'_{ij} \text{ in } L((C_n)_k^{\mathcal{P}}).$$

The adjacent vertices of v_i in C_n are v_{i-1} and v_{i+1} and so that there is no $C_n(C_n)_k^{\mathcal{P}}-$*path* from v_i to v_j if and only if v_{i-1} is not adjacent to v_j in $(C_n)_k^{\mathcal{P}}$, denoted as, $v_{i-1} \nsim v_j$ in $(C_n)_k^{\mathcal{P}}$ and similarly, $v_{i+1} \nsim v_j$ in $(C_n)_k^{\mathcal{P}}$, which means that,

$$l'_{i-1j} = l'_{i+1j} = 0, \quad \text{and hence,} \quad l'_{i-1j} + l'_{i+1j} = 0.$$

From Lemma 3.1,

$$l'_{ij-1} + l'_{ij+1} = 0, \quad \text{and thus,} \quad l'_{ij-1} = l'_{ij+1} = 0.$$

As a result, $v_i \nsim v_{j-1}$ and $v_i \nsim v_{j+1}$ in $(C_n)_k^{\mathcal{P}}$. This indicates that there is no $(C_n)_k^{\mathcal{P}}C_n$-*path* connecting v_i to v_j.

Case (ii): The number of $C_n(C_n)_k^{\mathcal{P}}-$*paths* and $(C_n)_k^{\mathcal{P}}C_n-$*paths* from v_i to v_j is exactly one.
Assume there exists exactly one $C_n(C_n)_k^{\mathcal{P}}-$*path* from v_i to v_j.
Then, $v_i \sim v_{i-1} \sim v_j$ or $v_i \sim v_{i+1} \sim v_j$, as the adjacent vertices of v_i in C_n are v_{i-1} and v_{i+1}. This implies that, either,

$$l'_{i-1j} = -1, \quad \text{or} \quad l'_{i+1j} = -1,$$

which in turn, by Lemma 3.1, yields that either,

$$l'_{ij-1} = -1, \quad \text{or} \quad l'_{ij+1} = -1.$$

Hence it follows that, either $v_i \sim v_{j-1}$ or $v_i \sim v_{j+1}$ in $(C_n)_k^{\mathcal{P}}$ and accordingly, either $v_i \sim v_{j-1} \sim v_j$ or $v_i \sim v_{j+1} \sim v_j$ results in a $(C_n)_k^{\mathcal{P}}C_n-$*path* from v_i to v_j.

Case (iii): The number of $C_n(C_n)^{\mathcal{P}}_k$ and $(C_n)^{\mathcal{P}}_k C_n-$*paths* from v_i to v_j is 2.
Let there exist two $(C_n)^{\mathcal{P}}_k C_n-$*paths* from v_i to v_j. Then, there exist k_1 and k_2 such that, $v_i \sim v_{k_1} \sim v_j$ and $v_i \sim v_{k_2} \sim v_j$. Hence,

$$l'_{ik_1} = -1 \text{ and } l'_{ik_2} = -1, \text{ where } \{k_1, k_2\} = \{j-1, j+1\}.$$

Now, from Lemma 3.1 it follows that,

$$\text{if } l'_{ij-1} + l'_{ij+1} = -2, \text{ then } l'_{i-1j} + l'_{i+1j} = -2.$$

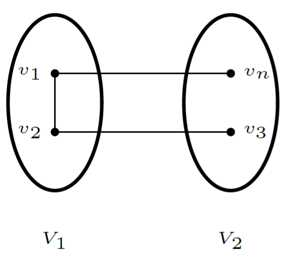
Thus, there exist two $C_n(C_n)^{\mathcal{P}}_k-$*paths* $v_i \sim v_{m_1} \sim v_j$ and $v_i \sim v_{m_2} \sim v_j$, where $m_1 \neq k_1$ and $m_2 \neq k_2$.

Consequently, in all three cases, the number of $C_n(C_n)^{\mathcal{P}}_k-$*paths* and $(C_n)^{\mathcal{P}}_k C_n-$*paths* from v_i to v_j are equal. □

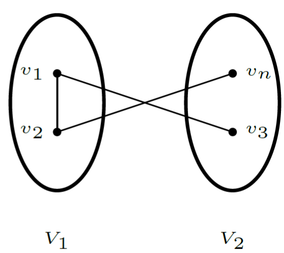
Theorem 3.6. *Let C_n be a Cycle graph on n vertices $\{v_1, v_2, v_3, \ldots, v_n\}$ and $(C_n)^{\mathcal{P}}_k$ be the $k-$complement of C_n with respect to the partition $\mathcal{P} = (V_1, V_2, \ldots, V_k)$ of $V(C_n)$. Then, the induced sub-graphs $\{V_i\}$ of C_n with regard to the partition $\mathcal{P}$ are totally disconnected if C_n and $(C_n)^{\mathcal{P}}_k$ are Laplacian commutative.*

Proof. Let C_n and $(C_n)^{\mathcal{P}}_k$ be Laplacian commutative. We show that no two vertices in V_i are adjacent for $i = 1, 2, \ldots, k$. Or equivalently, it is enough to show that, if $u \sim_{C_n} v$, then u and v belong to the same V_i.

Let $u \sim v$ in C_n, say $u = v_1$ and $v = v_2$. Suppose v_3 and v_n belong to the same partite set, say v_3 and $v_n \in V_2 \neq V_1$, as in Figure 2. Then, $v_n \sim v_1 \sim v_2$ is a $C_n(C_n)^{\mathcal{P}}_k-$*path* from v_n to v_2 through v_1, but from v_n to v_2 there exists no $(C_n)^{\mathcal{P}}_k C_n-$*path* because vertex v_{n-1} is not adjacent to v_2 in C_n and v_1 and v_3 are the only vertices adjacent to v_2 in C_n. If v_1, v_2, v_3, v_n all belong to same partite set, then also there exists two $C_n(C_n)^{\mathcal{P}}_k-$*paths* from v_n to v_2 through v_1 and v_{n-1} but only one $(C_n)^{\mathcal{P}}_k C_n-$*path* from v_n to v_2.



(a) Adjacency in C_n



(b) Adjacency in $(C_n)^{\mathcal{P}}_k$

Figure 2: Illustration for the proof of Theorem 3.6. Case: $v_3, v_n \in V_2$.

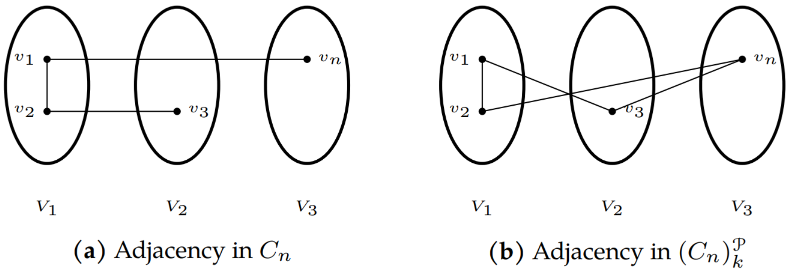


Figure 3: Illustration for the proof of Theorem 3.6. Case: $v_3 \in V_2, v_n \in V_3$.

Therefore, v_3 and v_n cannot belong to the same set.

Let $v_3 \in V_2$ and $v_n \in V_3$ as in Figure 3. Then, there exists a $C_n(C_n)_k^{\mathcal{P}}$*−path* from v_1 to v_n through v_2 and in order to get a $(C_n)_k^{\mathcal{P}}C_n$ path from v_1 to v_n, the vertex v_{n-1} cannot be in V_1.

The cases v_{n-1} belongs to V_2 or V_3 or a fourth partite set V_4 are described through the following figures,

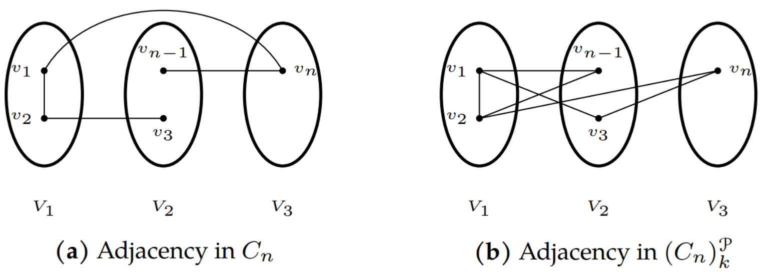


Figure 4: Illustration for the proof of Theorem 3.6. Case: $v_{n-1} \in V_2$.

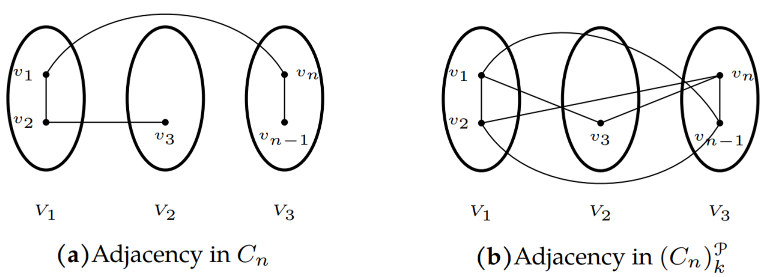
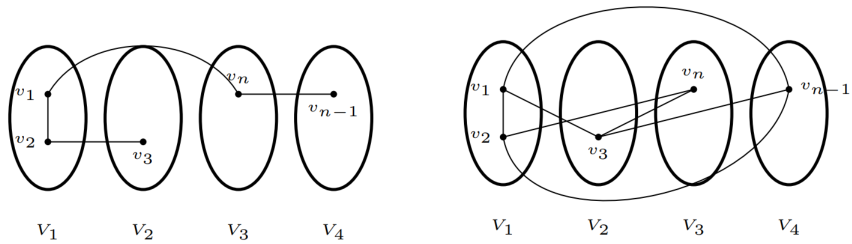


Figure 5: Illustration for the proof of Theorem 3.6. Case: $v_{n-1} \in V_3$.



(a) Adjacency in C_n (b) Adjacency in $(C_n)_k^{\mathcal{P}}$

Figure 6: Illustration for the proof of Theorem 3.6. Case: $v_{n-1} \in V_4$.

In Figure 4, the vertex $v_{n-1} \in V_2$ and hence, the number of $C_n(C_n)_k^{\mathcal{P}}-$*paths* from v_n to v_2 is two and $(C_n)_k^{\mathcal{P}}C_n-$*path* from v_n to v_2 is one. Similarly, Figures 5 and 6 also show that, the number of $C_n(C_n)_k^{\mathcal{P}}-$*paths* and $(C_n)_k^{\mathcal{P}}C_n-$*path* from v_n to v_2 are different, which is not possible by Theorem 3.5. Hence, when $L(C_n)$ commutes with $L(C_n)_k^{\mathcal{P}}$, no two adjacent vertices lie in the same partite set and hence, the induced sub-graphs $\{V_i\}$ with regard to a given partition $\mathcal{P}$ is totally disconnected. □

Theorem 3.7. *Let C_n be the Cycle graph on n vertices. Let $\mathcal{P}$ be a partition of the vertex set $V(C_n)$ and let $(C_n)_k^{\mathcal{P}}$ represent the $k-$complement of C_n concerning the partition $\mathcal{P}$. If the Laplacian matrices of the graphs C_n and $(C_n)_k^{\mathcal{P}}$ commute with each other, then for each i, any two vertices u and v belonging to the same set, V_i has identical $o-$degree in both C_n and $(C_n)_k^{\mathcal{P}}$.*

Proof. We have that the graphs C_n and $(C_n)_k^{\mathcal{P}}$ satisfy Laplacian commutativity. Then, $L(C_n)$ and $L((C_n)_k^{\mathcal{P}})$ commute with each other. Let $u, v \in V_i$, $1 \leq i \leq k-1$. Then from Theorem 3.5, the number of $C_n(C_n)_k^{\mathcal{P}}$ paths from u to v and the number of $(C_n)_k^{\mathcal{P}}C_n$ paths from u to v are equal. Also by Theorem 3.6, $u \nsim v$.

Since no two vertices of V_i can be adjacent, to prove the $o-degree$ of u equals $o-degree$ of v in $(C_n)_k^{\mathcal{P}}$, it is enough to prove that corresponding to each vertex x, adjacent to u, in $(C_n)_k^{\mathcal{P}}$, there exist a vertex y adjacent to v in $(C_n)_k^{\mathcal{P}}$ and vice-versa.

Let, $x \sim u$ in $_{(C_n)_k^{\mathcal{P}}}$. Which implies that $x \notin V_i$. (i.e., $x \in V_j, j \neq i$).

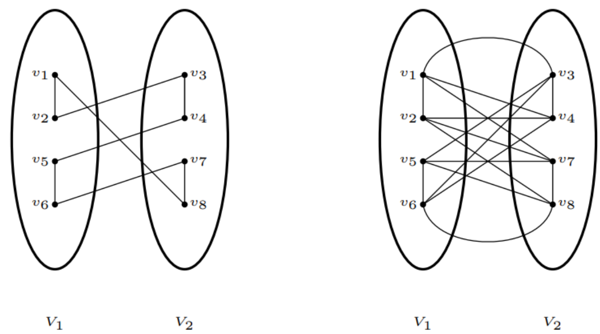
If $x \sim_{C_n} v$, then, $u \sim x \sim v$ gives the path $C_n(C_n)_k^{\mathcal{P}}$ from u to v through x. Then, there exists a $C_n(C_n)_k^{\mathcal{P}}$ path $u \sim y \sim v$, from u to v through y proves the claim.

If $x \nsim v$ in C_n, then, being in different partite sets, $x \sim v$ in $(C_n)_k^{\mathcal{P}}$ proves the claim.

Similarly, corresponding to each vertex adjacent to v, there exists a vertex adjacent to u in $(C_n)_k^{\mathcal{P}}$. □

Converse of the above theorem can not be true. The graphs are not necessarily Laplacian commutative simply because of identical out degree of vertices in V_i.

For example, consider the following partition of Cycle graph, C_8 as in the Figure 7 shown below. $V_1 = \{v_1, v_2, v_5, v_6\}$ and $V_2 = (v_3, v_4, v_7, v_8\}$.



(a). C_8 (b). $(C_8)^{\mathcal{P}}_2$

Figure 7: A partition of C_8 showing the converse of Theorem 3.7 is not true.

Here, $o-degree$ of each vertex v_i is 3, but with respect to the above partition, $L(C_n)$ and $L(C_n)^{\mathcal{P}}_k$ do not commute. Hence, it is also evident that the vertices in each partite set have equal $o-degree$ and the partite sets have equal cardinality, which will not be sufficient to provide Laplacian commutativity of C_n and $(C_n)^{\mathcal{P}}_k$. The next theorem explores three conditions to be satisfied by the partition $\mathcal{P}$ which altogether are sufficient for C_n and $(C_n)^{\mathcal{P}}_k$ to be Laplacian commutative.

Theorem 3.8. *Let $C_n = v_1v_2\ldots v_nv_1$ be the Cycle graph on n vertices, where n is not a prime and let, $\mathcal{P} = (V_1, V_2, V_3, \ldots, V_k)$ be a partition of the vertex set $V(C_n)$. Then, the graphs C_n and $(C_n)^{\mathcal{P}}_k$ are Laplacian commutative if $\mathcal{P}$ fulfills the following conditions concurrently,*

- (i) *V_i, $i = 1, 2, \ldots, k$ are of same cardinality.*
- (ii) *no two adjacent vertices of C_n lie in the same partite set, and*
- (iii) *if $V_j = \{v_{j_1}, v_{j_2}, v_{j_3}, \ldots, v_{j_m}\}$, where $j_1 \leq j_2 \leq \ldots \leq j_m$, then the distance $d(v_{j_i}, v_{j_{i+1}})$ (which are ≥ 2) are same, for $1 \leq i \leq m$, where $j_{m+1} = j_1$.*

Proof. Assume that, the partition $\mathcal{P}$ meets the assumptions (i), (ii) and (iii).

From assumptions we can conclude that, the number of vertices that are not in V_i but are adjacent to both v_i and v_j is the same.

Also we get, the vertices in each partite sets are equal in number and no two adjacent vertices belong to the same partite set. Which implies that, the induced sub-graphs $< V_i >$ are totally disconnected. Also, the distance between each vertex in a partite set is a constant. Hence, the corresponding Laplacian matrices of the graph and it's $k-$complement can be partitioned into k square blocks of equal order $|V_i|$, the cardinality of V_i. So, $L((C_n)^{\mathcal{P}}_k)$ is a circulant block matrix with diagonal entries $n - n_k - 2$, where n_k denotes the degree of v_i in $(C_n)^{\mathcal{P}}_k$.

Also, the Laplacian matrix of a cycle graph is of the form,

$$L(C_n) = \begin{bmatrix} A & B & 0 & 0 & \cdots & 0 & B^T \\ B^T & A & B & 0 & \cdots & 0 & 0 \\ 0 & B^T & A & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B & 0 & 0 & 0 & \cdots & B^T & A \end{bmatrix}.$$

The product of two circulant matrices is circulant and they commute with each other. Hence, C_n and $(C_n)_k^{\mathcal{P}}$ are Laplacian commutative. □

4 Discussion

This paper introduces the Laplacian commutativity of different families of graphs. This study, in a broader aspect, has many limitations, so that the main focus is confined to the Laplacian commutativity of complete graphs, K_n and cycle graphs, C_n with their $k-$complement with respect to specific vertex partitions. By deriving some necessary and sufficient conditions, for the partition $\mathcal{P} = (V_1, V_2, V_3, \ldots, V_k)$ under which the Laplacian matrices of C_n and K_n commute, we have extended the understanding of graph theoretic properties through Laplacian matrices. By focusing on the structural properties of the cycle graph and its transformation into the $k-$complement, we have identified specific partitions that preserve the commutativity of the Laplacian matrices.

The Laplacian commutativity depends on the structure of the vertex partitions and the corresponding $k-$complement graph. More specifically, symmetry in the partition and the distribution of vertices play significant roles, and they often indicate underlying structural symmetry or alignment between two graphs, revealing deeper relationships between their connectivity patterns. It provides insights into how graph partitions or modifications influence spectral and structural properties. The results provide a framework for exploring commutativity in broader families of graphs and graph operations.

Future work could explore generalizing these results to broader graph families and analyzing their implications in dynamic and higher$-$dimensional graph systems, and could study the impact of Laplacian commutativity on the systems where Laplacian matrices play crucial roles.

5 Conclusions

This study adds to the knowledge in spectral graph theory and opens new paths for exploring graph operations and their effects on spectra. If two Laplacian matrices commute, they can be simultaneously diagonalized, meaning their eigenvalues and eigenvectors can be computed together, which significantly simplifies spectral computations, especially for analyzing graph products or combinations.

Acknowledgment We thank the editor and the reviewers for their constructive comments and suggestions. The first author gratefully acknowledges the financial support provided by the Council of Scientific and Industrial Research (CSIR), Government of India.

Conflicts of Interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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